Rewarding Learning

ADVANCED
General Certificate of Education
2022 Reserve Series

Further Mathematics

Assessment Unit A2 1
assessing
Pure Mathematics

[AFM11]

WEDNESDAY 29 JUNE, AFTERNOON

MARK SCHEME

GCE ADVANCED/ADVANCED SUBSIDIARY (AS) FURTHER MATHEMATICS

Introduction
The mark scheme normally provides the most popular solution to each question. Other solutions given by candidates are evaluated and credit given as appropriate; these alternative methods are not usually illustrated in the published mark scheme.

The marks awarded for each question are shown in the right-hand column and they are prefixed by the letters **M**, **W** and **MW** as appropriate. The key to the mark scheme is given below:

M indicates marks for correct method.

W indicates marks for working.

MW indicates marks for combined method and working.

The solution to a question gains marks for correct method and marks for an accurate working based on this method. Where the method is not correct no marks can be given.

A later part of a question may require a candidate to use an answer obtained from an earlier part of the same question. A candidate who gets the wrong answer to the earlier part and goes on to the later part is naturally unaware that the wrong data is being used and is actually undertaking the solution of a parallel problem from the point at which the error occurred. If such a candidate continues to apply correct method, then the candidate's individual working must be followed through from the error. If no further errors are made, then the candidate is penalised only for the initial error. Solutions containing two or more working or transcription errors are treated in the same way. This process is usually referred to as "follow-through marking" and allows a candidate to gain credit for that part of a solution which follows a working or transcription error.

Positive marking:
It is our intention to reward candidates for any demonstration of relevant knowledge, skills or understanding. For this reason we adopt a policy of **following through** their answers, that is, having penalised a candidate for an error, we mark the succeeding parts of the question using the candidate's value or answers and award marks accordingly.

Some common examples of this occur in the following cases:

(a) a numerical error in one entry in a table of values might lead to several answers being incorrect, but these might not be essentially separate errors;

(b) readings taken from candidates' inaccurate graphs may not agree with the answers expected but might be consistent with the graphs drawn.

When the candidate misreads a question in such a way as to make the question easier only a proportion of the marks will be available (based on the professional judgement of the examining team).

COVID-19 Context
Given the unprecedented circumstances presented by the COVID-19 public health crisis, senior examiners, under the instruction of CCEA awarding organisation, are required to train assistant examiners to apply the mark scheme in case of disrupted learning and lost teaching time. The interpretation and intended application of the mark scheme for this examination series will be communicated through the standardising meeting by the Chief or Principal Examiner and will be monitored through the supervision period. This paragraph will apply to examination series in 2021-2022 only.

AVAILABLE MARKS

1 $\displaystyle\sum_{r=n+1}^{2n} r(4r-1)(4r+1) \equiv \sum_{r=n+1}^{2n}\left(16r^3 - r\right)$ MW1

$\displaystyle\equiv \sum_{r=1}^{2n}\left(16r^3 - r\right) - \sum_{r=1}^{n}\left(16r^3 - r\right)$ M1 W1

$\displaystyle\equiv \sum_{r=1}^{2n}16r^3 - \sum_{r=1}^{2n}r - \sum_{r=1}^{n}16r^3 + \sum_{r=1}^{n}r$ MW1

$\displaystyle\equiv 16 \times \frac{1}{4}(2n)^2(2n+1)^2 - \frac{1}{2} \times 2n(2n+1) - 16 \times \frac{1}{4}n^2(n+1)^2 + \frac{1}{2}n(n+1)$ M2W2

$\displaystyle\equiv 16n^2(2n+1)^2 - n(2n+1) - 4n^2(n+1)^2 + \frac{1}{2}n(n+1)$

$\displaystyle\equiv 64n^4 + 64n^3 + 16n^2 - 2n^2 - n - 4n^4 - 8n^3 - 4n^2 + \frac{1}{2}n^2 + \frac{1}{2}n$

$\displaystyle\equiv 60n^4 + 56n^3 + \frac{21}{2}n^2 - \frac{1}{2}n$ MW1 9

			AVAILABLE MARKS
2	Test for $n = 1$		
	$\sum_{r=1}^{1} \frac{1}{4r^2 - 1} = \frac{1}{4-1} = \frac{1}{3}$	M1	
	$\frac{n}{2n+1} \rightarrow \frac{1}{2(1)+1} = \frac{1}{3}$		
	Hence true for $n = 1$	W1	
	Assume true for $n = k$		
	$\Rightarrow \sum_{r=1}^{k} \frac{1}{4r^2 - 1} \equiv \frac{k}{2k+1}$	MW1	
	Test for $n = k + 1$		
	$\Rightarrow \sum_{r=1}^{k+1} \frac{1}{4r^2 - 1} \equiv \frac{k}{2k+1} + \frac{1}{4(k+1)^2 - 1}$	M1W1	
	$\equiv \frac{k}{2k+1} + \frac{1}{4k^2 + 8k + 3}$		
	$\equiv \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$		
	$\equiv \frac{k(2k+3)+1}{(2k+1)(2k+3)}$		
	$\equiv \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$		
	$\equiv \frac{(k+1)(2k+1)}{(2k+1)(2k+3)}$		
	$\equiv \frac{k+1}{2(k+1)+1}$	W1	
	$\Rightarrow$ True for $n = k + 1$	W1	
	Since true for $n = 1$, then by induction this is true for all $n \geqslant 1$	MW1	8

AVAILABLE MARKS

3 $\int_1^\infty \left(\frac{1}{x+1} - \frac{1}{x}\right)\mathrm{d}x$

$\Rightarrow \lim_{t\to\infty}\int_1^t \left(\frac{1}{x+1} - \frac{1}{x}\right)\mathrm{d}x$ MW1

$= \lim_{t\to\infty}[\ln|x+1| - \ln|x|]_1^t$ MW1

$= \lim_{t\to\infty}\left[\ln\left|\frac{x+1}{x}\right|\right]_1^t$ MW1

$= \lim_{t\to\infty}\left[\ln\left|\frac{t+1}{t}\right| - \ln 2\right] = \lim_{t\to\infty}\left[\ln\left|1+\frac{1}{t}\right| - \ln 2\right]$ M1

$= \ln 1 - \ln 2$

$= -\ln 2$ W1 5

4 $A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,\mathrm{d}\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (2+\sin\theta)^2\,\mathrm{d}\theta$ M1 W2

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (4 + 4\sin\theta + \sin^2\theta)\,\mathrm{d}\theta$ MW1

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(4 + 4\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right)\mathrm{d}\theta$ M1W1

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{9}{2} + 4\sin\theta - \frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$

$= \frac{1}{2}\left[\frac{9\theta}{2} - 4\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$ M1W1

$= \frac{1}{2}\left[\frac{9\pi}{4} + 4\right]$

$= \left[\frac{9\pi}{8} + 2\right]$ square units MW1 9

5 Integrating factor

AVAILABLE MARKS

$= e^{\int 2\,dx}$ M1

$= e^{2x}$ W1

Multiply through by IF

$\Rightarrow e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}\sinh x$ MW1

$\Rightarrow \frac{d}{dx}(e^{2x}y) = e^{2x}\sinh x$

Integrate

$\Rightarrow e^{2x}y = \int e^{2x}\sinh x\,dx$ M1

$e^{2x}y = \frac{1}{2}\int e^{2x}(e^{x} - e^{-x})\,dx$ M1

$e^{2x}y = \frac{1}{2}\int (e^{3x} - e^{x})\,dx$ W1

$e^{2x}y = \frac{1}{2}\left(\frac{1}{3}e^{3x} - e^{x}\right) + c$ MW1

$y = \frac{1}{6}e^{x} - \frac{1}{2}e^{-x} + ce^{-2x}$ W1

$x = 0, y = 0 \Rightarrow 0 = \frac{1}{6} - \frac{1}{2} + c$ M1

$c = \frac{1}{3}$

$\Rightarrow\ y = \frac{1}{6}e^{x} - \frac{1}{2}e^{-x} + \frac{1}{3}e^{-2x}$ W1 10

				AVAILABLE MARKS
6	(i)	$\dfrac{7-x}{(1+x^2)(1-3x)} \equiv \dfrac{Ax+B}{1+x^2} + \dfrac{C}{1-3x}$	M1	
		Compare numerators		
		$\Rightarrow (Ax+B)(1-3x) + C(1+x^2) \equiv 7-x$	M1	
		$x = \dfrac{1}{3} \quad \Rightarrow \dfrac{10}{9}C = \dfrac{20}{3}$	M1	
		$C = 6$	W1	
		x^2 term $\quad \Rightarrow -3A + C = 0$		
		$A = 2$		
		Constant $\Rightarrow B + C = 7$		
		$B = 1$	MW1	
		$\Rightarrow \dfrac{7-x}{(1+x^2)(1-3x)} \equiv \dfrac{2x+1}{1+x^2} + \dfrac{6}{1-3x}$		
	(ii)	$(2x+1)(1+x^2)^{-1} = (2x+1)\left(1 + (-1)x^2 + \dfrac{(-1)(-2)}{2!}x^4 + \ldots\right)$	M1 W1	
		$= 2x - 2x^3 + 1 - x^2 + x^4 + \ldots$		
		$= 1 + 2x - x^2 - 2x^3 + x^4 + \ldots$	MW1	
		$6(1-3x)^{-1} = 6\left(1 + (-1)(-3x) + \dfrac{(-1)(-2)}{2!}(-3x)^2\right.$		
		$\left. + \dfrac{(-1)(-2)(-3)}{3!}(-3x)^3 + \dfrac{(-1)(-2)(-3)(-4)}{4!}(-3x)^4 + \ldots\right)$	M1	
		$= 6(1 + 3x + 9x^2 + 27x^3 + 81x^4 + \ldots)$		
		$= 6 + 18x + 54x^2 + 162x^3 + 486x^4 + \ldots$	W1	
		$\Rightarrow \dfrac{7-x}{(1+x^2)(1-3x)} = 7 + 20x + 53x^2 + 160x^3 + 487x^4 + \ldots$	M1 W1	
	(iii)	$\lvert x^2 \rvert < 1 \ \Rightarrow \ -1 < x < 1$	MW1	
		$\lvert -3x \rvert < 1 \Rightarrow -\dfrac{1}{3} < x < \dfrac{1}{3}$	MW1	
		Hence the range for the complete series is $-\dfrac{1}{3} < x < \dfrac{1}{3}$	W1	15

7	$\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 8y = 16x^2 + 2$		**AVAILABLE MARKS**
	Auxiliary Equation		
	$m^2 - 2m - 8 = 0$	M1	
	$(m - 4)(m + 2) = 0$		
	$\Rightarrow m = 4,\ m = -2$	W1	
	$\Rightarrow y = Ae^{4x} + Be^{-2x}$	M1 W1	
	Particular Integral		
	$y = Cx^2 + Dx + E$	M1	
	$\dfrac{dy}{dx} = 2Cx + D$	MW1	
	$\dfrac{d^2y}{dx^2} = 2C$	MW1	
	$\Rightarrow 2C - 2(2Cx + D) - 8(Cx^2 + Dx + E) \equiv 16x^2 + 2$	M1 W1	
	Compare coefficients		
	x^2 : $-8C = 16 \quad \Rightarrow C = -2$	M1 W1	
	x : $-4C - 8D = 0 \quad \Rightarrow D = 1$		
	Constant: $2C - 2D - 8E = 2$		
	$-4 - 2 - 8E = 2$		
	$E = -1$	W1	
	General Solution		
	$y = Ae^{4x} + Be^{-2x} - 2x^2 + x - 1$	W1	13

AVAILABLE MARKS

8 (a) (i) $f(x) = (1 + x^2)\sin x$

$f'(x) = (1 + x^2)\cos x + 2x \sin x$ M1 W1

$f''(x) = -(1 + x^2)\sin x + 2x\cos x + 2\sin x + 2x\cos x$

$= \sin x - x^2 \sin x + 4x\cos x$ MW2

(ii) $f'''(x) = \cos x - 2x\sin x - x^2\cos x + 4\cos x - 4x\sin x$

$= 5\cos x - 6x\sin x - x^2\cos x$ MW1

$f(0) = 0 \quad f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = 5$ M1 W1

$\Rightarrow \quad f(x) \equiv 0 + 1x + 0x^2 + \frac{5}{3!}x^3 + \ldots$ M1

$\Rightarrow \quad f(x) \equiv x + \frac{5x^3}{6} + \ldots$ W1

(b) $\sin x \approx x$

$\cos x \approx 1 - \frac{1}{2}x^2$ MW1

$30x + 400\left(1 - \frac{1}{2}x^2\right) = 401$ M1 W1

$200x^2 - 30x + 1 = 0$ W1

$x = \frac{30 \pm \sqrt{900 - 800}}{400}$

$x = 0.05, x = 0.1$ W1 14

AVAILABLE MARKS

9 **(a)** $y = \sin^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) + \ln(1+x^2)$

$$\frac{dy}{dx} = \frac{-\frac{1}{2} \times 2x(1+x^2)^{-\frac{3}{2}}}{\sqrt{1-\frac{1}{1+x^2}}} + \frac{2x}{1+x^2}$$ M2 W1

MW1

$$= \frac{-x}{(1+x^2)\sqrt{1+x^2-1}} + \frac{2x}{1+x^2}$$ MW1

$$= \frac{-1}{1+x^2} + \frac{2x}{1+x^2}$$

$$= \frac{2x-1}{1+x^2}$$ W1

$$\frac{dy}{dx} = 0 \Rightarrow \frac{2x-1}{1+x^2} = 0$$ M1

$$\Rightarrow x = \frac{1}{2}$$ W1

Hence there is only one stationary point. MW1

(b) $u = x + 1$

$$\frac{du}{dx} = 1$$ MW1

$x = -1,\ u = 0$

$x = 2,\ u = 3$ MW1

$$\Rightarrow \int_0^3 \frac{du}{4(u-1)^2 + 8(u-1) + 13}$$ M1 W1

$$= \int_0^3 \frac{du}{4u^2 + 9}$$ W1

$$= \frac{1}{4}\int_0^3 \frac{du}{u^2 + 9/4}$$

$$= \frac{1}{4} \times \frac{1}{\left(\frac{3}{2}\right)}\left[\tan^{-1}\left(\frac{2u}{3}\right)\right]_0^3$$ M1 W1

$$= \frac{1}{6}\left[\tan^{-1}(2) - \tan^{-1}(0)\right]$$ M1

$$= \frac{1}{6}\tan^{-1}(2)$$ W1 18

		AVAILABLE MARKS
10 **(i)** $I_n = \int_0^1 x^n (1-x)^{\frac{1}{2}}\, dx$		
$u = x^n \qquad \dfrac{dv}{dx} = (1-x)^{\frac{1}{2}}$	M1 W1	
$\dfrac{du}{dx} = nx^{n-1} \qquad v = -\dfrac{2}{3}(1-x)^{\frac{3}{2}}$	MW2	
$\Rightarrow I_n = \left[-\dfrac{2}{3}x^n(1-x)^{\frac{3}{2}}\right]_0^1 - \int_0^1 -\dfrac{2}{3}nx^{n-1}(1-x)^{\frac{3}{2}}\, dx$	MW1	
$I_n = 0 + \dfrac{2}{3}n\int_0^1 x^{n-1}(1-x)(1-x)^{\frac{1}{2}}\, dx$	MW1 M1	
$I_n = 0 + \dfrac{2}{3}n\int_0^1 x^{n-1}(1-x)^{\frac{1}{2}}\, dx - \dfrac{2}{3}n\int_0^1 x^n(1-x)^{\frac{1}{2}}\, dx$	W1	
$I_n = \dfrac{2}{3}nI_{n-1} - \dfrac{2}{3}nI_n$	MW1	
$I_n\left(1 + \dfrac{2}{3}n\right) = \dfrac{2}{3}nI_{n-1}$		
$I_n = \dfrac{2n}{2n+3}I_{n-1}$	W1	
(ii) Graph cuts x-axis $\Rightarrow x^2(1-x)^{\frac{1}{2}} = 0$		
$\Rightarrow x = 0, 1$	MW1	
Area $= \int_0^1 x^2(1-x)^{\frac{1}{2}}\, dx$	M1	
Area $= I_2$	M1	
$I_0 = \int_0^1 (1-x)^{\frac{1}{2}}\, dx$	M1	
$= \left[-\dfrac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1$		
$= 0 - \left(-\dfrac{2}{3}\right)$		
$= \dfrac{2}{3}$	W1	
$I_1 = \dfrac{2}{5}I_0 = \dfrac{4}{15}$	MW1	
$I_2 = \dfrac{4}{7}I_0 = \dfrac{16}{105}$		
Area $= \dfrac{16}{105}$ square units	MW1	17

				AVAILABLE MARKS
11	**(i)**	$\cosh y = x$	MW1	
		$\Rightarrow \sinh y \dfrac{dy}{dx} = 1$	M1 W1	
		$\dfrac{dy}{dx} = \dfrac{1}{\sinh y}$		
		$= \dfrac{1}{\sqrt{\cosh^2 y - 1}}$	MW1	
		$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 - 1}}$	MW1	
	(ii)	$25x^2 + 20x + 3 \equiv 25x^2 + 20x + 4 - 1$		
		$\equiv (5x + 2)^2 - 1$	MW1	
	(iii)	$\displaystyle\int \frac{10x + 5}{\sqrt{25x^2 + 20x + 3}}\,dx$		
		$\displaystyle= \int \frac{1}{\sqrt{25x^2 + 20x + 3}}\,dx + \int \frac{10x + 4}{\sqrt{25x^2 + 20x + 3}}\,dx$	M1 W1	
		$\displaystyle= \int \frac{1}{\sqrt{(5x + 2)^2 - 1}}\,dx + \int \frac{10x + 4}{\sqrt{25x^2 + 20x + 3}}\,dx$		
		Consider $\displaystyle\int \frac{1}{\sqrt{(5x + 2)^2 - 1}}\,dx$		
		Let $u = 5x + 2$	M1	
		$\Rightarrow \dfrac{du}{dx} = 5$		
		$\displaystyle\Rightarrow \int \frac{du}{5\sqrt{u^2 - 1}}$	MW1	
		$= \dfrac{1}{5} \cosh^{-1} u$		
		$= \dfrac{1}{5} \cosh^{-1} (5x + 2)$	W1	
		$\displaystyle\int \frac{10x + 4}{\sqrt{25x^2 + 20x + 3}}\,dx = \frac{1}{5} \times 2\sqrt{25x^2 + 20x + 3}$	M1 W1	
		$\displaystyle\Rightarrow \int \frac{10x + 5}{\sqrt{25x^2 + 20x + 3}}\,dx = \frac{1}{5}\left[\cosh^{-1} (5x + 2) + 2\sqrt{25x^2 + 20x + 3}\right] + c$	MW1	14

		AVAILABLE MARKS
12 **(i)** $\Rightarrow z = \cos\theta + \mathrm{i}\sin\theta$		
$\Rightarrow \frac{1}{z} = (\cos\theta + \mathrm{i}\sin\theta)^{-1}$	MW1	
$\frac{1}{z} = \cos\theta - \mathrm{i}\sin\theta$		
$\Rightarrow z - \frac{1}{z} = 2\mathrm{i}\sin\theta$	M1 W1	
(ii) $\left(z - \frac{1}{z}\right)^6 = 2^6\mathrm{i}^6\sin^6\theta$	M1	
$\left(z - \frac{1}{z}\right)^6 = z^6 + 6z^5\left(-\frac{1}{z}\right) + 15z^4\left(-\frac{1}{z}\right)^2 + 20z^3\left(-\frac{1}{z}\right)^3$	M1	
$+ 15z^2\left(-\frac{1}{z}\right)^4 + 6z\left(-\frac{1}{z}\right)^5 + \left(-\frac{1}{z}\right)^6$	W1	
$= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$	MW1	
But $z + \frac{1}{z} = 2\cos\theta \Rightarrow z^n + \frac{1}{z^n} = 2\cos n\theta$	M1 W1	
$\Rightarrow \left(z - \frac{1}{z}\right)^6 = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$	MW1	
$\Rightarrow -64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$	MW1	
$\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$	MW1	
(iii) $2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta = 19$		
$2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20 = -1$	MW1	
$\Rightarrow -2 \times 32\sin^6\theta = -1$	M1 W1	
$64\sin^6\theta = 1$		
$\sin^6\theta = \frac{1}{64}$	W1	
$\sin\theta = \pm\frac{1}{2}$		
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$	MW2	18
	Total	**150**